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CRITICISMS AND DISCUSSIONS.

THE NATURE AND VALIDITY OF THE PRINCIPLE OF LEAST ACTION.

This paper is critical and expository. I have already* examined, in some detail, the views of Maupertuis, Euler, d'Arcy, and others: we shall now be concerned with testing these views. The object of this testing is what I take to be the object of all historical and critical investigation in science: the elucidation of principles and methods by emphasis on what has shown itself to be psychologically important. In science and in the philosophy of science we are concerned with truths of fact, and usually only very indirectly with the people who assert or deny these truths. But, while this is our object, we know by experience that we can get a firm grasp of the truth by studying the aims, mistakes, and methods of those who have sought and found these truths. This is, perhaps, the only way of getting a firm grasp of the truths. But this is a psychological question. It is, I think, a defect in human nature that different values are placed on different truths; all truths should be of equal value. History and criticism often teach us to counteract this wrong apportionment of values by drawing our attention repeatedly to overlooked or underestimated truth. For example, the principle of least action (A) necessarily requires a certain condition which we will call B. Now B seems to require C (the conservation of *vis viva*) to hold, and C certainly implies B. But B does not imply C, though many very eminent mathematicians have thought that it does. They never explicitly formulated this opinion, or else of course they would have seen its error.[1] Now history and criticism lead us *to lay stress on* the facts that: (1) A implies B and B implies A; (2) C implies B, but B does not imply C; (3) C implies A, but A does not imply C. On the other hand, if we merely write the theorem in the form: C implies B, B implies A, and A implies B, we lose sight of the important fact that B does not imply C. The fact that B does not imply C might, indeed, be stated in logical terms, but, as it is what is called a "negative" proposition and gives rise to no "useful" conclusions, it is omitted from most text-books.

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* *Monist*, April and July, 1912, Vol. XXII, pp. 285-304, 414-459.

[1] I have heard of a lecturer who used to tell his pupils not to bring their difficulties to him until they could express them clearly. He might almost as well have told them always to settle their own difficulties for themselves. Often the whole difficulty of a difficulty lies in our not being able to set it down in words, because our ideas are vague or confused, often unconsciously so.

But our considerations are not only about text-books, which only too often succeed in making an unfortunate compromise between logic and history. We must emphasize that a perfect *logical* statement of truths is often of no more help to us than an ordinary text-book. Prolixity is an enemy to our process of understanding, but it is often necessary in logic; redundancy is never necessary in logic, but is often necessary to the correct working of our process of understanding. We know the value of judicious repetition when teaching, and historical and critical studies are concerned with the teaching of truths. Of course we are liable to fall into the error of saying, and perhaps thinking, that the truths with which it is our object to become acquainted *develop.* But it is our knowledge of them that develops. Some philosophers think it is a demerit not to develop or be in motion, and so maintain that truth grows, or is fluid. But this seems as bad as confusion of the qualities of America with those of its discoverer.

In logic we say that C implies B; we do not need to emphasize that this does not imply that B implies C; and yet the history of the principle of least action shows us how important this emphasis may be. Then again, we know how important psychologically may be the platitude that such and such a thing may or may not be so. Then too there is the fact that in a purely logical exposition the definitions, though logically irrelevant, are important.[2]

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[2] A. N. Whitehead and B. Russell, *Principia Mathematica,* Vol. I, Cambridge, 1910, p. 12: "In spite of the fact that definitions are theoretically superfluous, it is nevertheless true that they often convey more important information than is contained in the propositions in which they are used. This arises from two causes. First, a definition usually implies that the *definiens* is worthy of careful consideration. Hence the collection of definitions embodies our choice of subjects and our judgment as to what is most important. Secondly, when what is defined is (as often occurs) something already familiar, such as cardinal or ordinal numbers, the definition contains an analysis of a common idea, and may therefore express a notable advance. Cantor's definition of the continuum illustrates this: his definition amounts to the statement that what he is defining is the object which has the properties commonly associated with the word 'continuum,' though what precisely constitutes these properties had

We have already considered the historical question relating to the principle of least action, and may now—with some relief, considering the not wholly pleasant side of human nature which we have had to take into account—leave history and confine our attention to criticism. We shall examine into the validity of the various views we have described with some minuteness, making use, where necessary, of the results of modern investigations on the principles of least action and allied principles. At the close, I trust that we shall have attained to a quite satisfactory conception of the principles in question, for a historical and critical research seems to me to present special advantages. In the first place, we see in work actual methods of investigation, and these examples are valuable guides, both in showing us the best sort of thing to do and what sort of thing to avoid; and, in the second place, we are, I repeat, led to emphasize the psychologically important points in a principle or theorem. In a theorem written out in a perfect logical form, a hypothesis on which it may be important psychologically to lay stress is not particularly marked off from the others, and a hypothesis which is not made, and which it is important to mention as not having been made, is simply unmentioned.

In the present article I will first of all give some considerations on the differential equations of mechanics, which will enable us to determine the nature of the principle of least action and fix its position with respect to other principles such as that of d'Arcy. Then we come to the far harder question as to the extent of the validity of the principle, but this has, I think, been answered quite satisfactorily in modern times, principally owing to the work of Hölder. I have previously dealt with this matter.

The second part (section IV) of the present article will, then, be devoted to an examination of the assertions made about and problems alleged to be solved by, the principle of least action.

I.

The differential equations of mechanics are *ordinary* equations of the *second order*; that is to say, there is only one independent variable—not many, as in the theory of the conduction of heat[3]—

not before been known. In such cases, a definition is a 'making definite': it gives definiteness to an idea which had previously been more or less vague." In 1894 Peano (*Notations de logique mathématique,* Turin, 1894, p. 49) said: "The utility of definitions is well known. But we must remark that, rigorously speaking, they are not necessary."

[3] Cf. Mach, *Die Principien der Wärmelehre,* 2d ed., Leipsic, 1900, pp. 82-100.

and the order of the highest differential quotients occurring in them is the second. The dynamical equations are of a simple form that offers special facilities for integration;[4] but even now these equations cannot be integrated generally. The typical form of such equations is, to begin with, the simple case of one particle of mass m acted upon by what is known as a "conservative" system of forces,

$$m(d^2x/dt^2) = X,$$

where X is a function of the coordinates (x, y, z) of the particle, and not of t.[5] There are three such equations (for the three coordinates) for each such particle m, and the other two are:

$$m(d^2y/dt^2) = Y, \quad m(d^2z/dt^2) = Z.$$

For a swarm of n such particles we have $3n$ equations. Here come in mutual attractions and so on, that we will consider below.

Of course such "particles"—mathematical points endowed with mass—are mathematical fictions, and only a first approximation to reality. But, building on such foundations, we reach, as Newton did and as students of the theory of attractions know well, a theorem according to which nothing is altered—possible impacts apart—if the particles are replaced by homogeneous spheres of matter. This gives a second, and much closer, approximation to the actual state of things in the solar system. This example, by the way, is typical of the manner in which we always proceed in mathematical physics. By successive attempts we make our mathematical model imitate actuality more and more, somewhat as a portrait-painter proceeds;

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[4] Chief among the works that deal with the integration of the dynamical equations are Jacobi's lectures on dynamics (*C. G. J. Jacobi's Vorlesungen über Dynamik. Gehalten an der Universität Königsberg im Wintersemester 1842-1843 und nach einem von C. W. Borchardt ausgearbeiteten Hefte herausgegeben von A. Clebsch,* Berlin, 1866; second edition, revised by. E. Lottner, in *C. G. J. Jacobis Gesammelte Werke*: *Supplementband,* Berlin, 1884.)

[5] It may not be amiss to mention here a difficulty with regard to the meaning of the dynamical equations which often puzzles students and, I regret to say, teachers too. It is this: "If $m.d^2x/dt^2$ is the force on the particle m, and X is also this force, what is the above equation but an identity? And is it maintained that an identity of the form $X = X$ is the fundamental principle of dynamics? The answer is that whereas (supposing, for brevity, that we have only one coordinate, x, to deal with) X is a known function of the unknown x, x is an unknown function of the unknown t. The above equation then asks the question: What function of t must x be in order that $m.d^2x/dt^2$ is identical with X? When this question is answered (when, as we say, the equation is "solved"), the only unknown remaining is t. That is to say, the function that x is of t is determined for a whole class (usually infinite) of values of t. This is what is meant by calling t a "variable." Of course, t and x and so on denote (ambiguously) the numbers which are the measures of the times, spaces, and so on, considered, and not these physical objects themselves. Yet, for the sake of shortness, I will say "v is the velocity" for "v is the numerical measure of the velocity."

but the portrait-painter, urged by artistic feeling, sometimes modifies, emphasizes, and suppresses, while we only consciously do these things when we are occupied with the psychological business of instilling truths discovered about nature into ourselves or others.

To return to the differential equations of motion. Let us consider a very simple case. Take the general equation of rectilinear motion in dynamics:

$$(d^2s/dt^2) = f(s),$$

where s is the distance measured from a fixed center of force upon the line of motion. Introducing the abbreviation v for the velocity, the above equation turns into a pair of differential equations of the first order, for s and v:

$$(dv/dt) = f(s), \quad (ds/dt) = v.$$

Now use the infinitesimal method[6] and eliminate, by ordinary algebra, dt from this pair of equations. We get:

$$v\,.\,dv = f(s)\,.\,ds,$$

which can be immediately integrated in the form:

$$\tfrac{1}{2}v = \int f(s)\,.\,ds + \text{const.}$$

Then we substitute the value of v we get from this in the equations of the pair, and get s in terms of t and t in terms of s. The result of the first integration is called a "first integral" of the given equation of the second order; it contains one constant of integration, and its complete integral, which contains a second constant, is the "complete integral" of the given equation. The complete integral of an equation of the second order is a relation between the independent variable, the dependent variable (in our typical case, t and s respectively), and two arbitrary constants; and conversely, any such relation may be regarded as a primitive, from which a differential equation free from arbitrary constants can be obtained.

Let us consider how the first integral referred to gradually appeared and took its place in the history of mechanics.

Galileo considered the motion of a single body near the surface of the earth, and it was Huygens who first published an account of a mechanical problem involving many bodies. It cannot, by the way, be emphasized too strongly that it was Galileo's researches alone which were used in Newton's early work. Galileo, by his discovery of the law of inertia, had destroyed the old belief that motion always requires the continual exertion of some force to keep it up. Thus, to explain the motions of the planets, was required, not the vortices of Descartes, but some force by which the planet was continually deflected from its straight and uniformly described course. Newton began by applying the dynamical principles of Galileo to astronomical events. When he came to consider the mutual actions and motions of two bodies, it is probable that, by formulating an instinctive belief of himself and others, he had attained his fundamental law before Huygens's results were printed. Newton's researches were due to three influences: first and foremost were Galileo's dynamical investigations; then there was Copernicus's (1473-1543) revival of the theory that the sun is the center of our system; and lastly there were the laws discovered by Johann Kepler (1571-1630) as a result of long study of the astronomical observations of his teacher Tycho Brahe (1546-1601) and himself.

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[6] On another occasion, I will discuss the legitimacy of treating, as is universally done in analytical physics, dx and dt as if they were small, finite, real numbers.

Returning to the history of the above integral, it will be instructive to sum up what Huygens[7] did. Galileo[8] had discovered three equations holding between the time during which a body falls near the surface of the earth, the distance fallen through, and the velocity attained by the falling. One of these—the one from which Galileo set out—is

$$v = gt,$$

and in modern terminology, since $v = ds/dt$, we can get Galileo's relation between s and t by integration. From this, $s = gt^2/2$, and the first relation, we get Galileo's equation between s and v:

$$gs = v^2/2.$$

Now, this last equation is a first integral of the differential system:

$$dv/dt = g, \quad ds/dt = v,$$

for falling motion near the surface of the earth; for, if we differentiate the last equation but two, the result becomes an identity if we substitute g for dv/dt, as we are allowed to do by the given system.

If we introduce the conception of *mass*, we can put Huygens's generalization of Galileo's integral into the form:[9]

$$\Sigma ph = \tfrac{1}{2}\Sigma mv^2;$$

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[7] Mach, *Mechanics*, pp. 173-180.

[8] *Ibid.*, p. 148.

[9] Mach, *Mechanics*, p. 178.

and this again, when the forces are not necessarily constant nor the paths of the masses rectilinear,[10] becomes

$$\Sigma\int p.ds=\tfrac{1}{2}\Sigma m(v^2-v_0^2),$$

or, as it is usually written when a force-function U exists,

$$U-U_0=T-T_0.$$

This generalization is, we can say, immediately brought about by familiarity with the infinitesimal calculus.

Now, this last equation is a first integral of the system of dynamical equations, just as, in the above case, Galileo's equation is a first integral of the system of equations of the second order which describes the motion of falling. That is to say, if we differentiate the equation between T and U, the result becomes an identity by means of the relations given by the general equations of dynamics. The integral in question subsists under certain conditions which a precise mathematical formulation reveals. We will come on this question again immediately and also farther on.

When, as in the case of the compound pendulum, there is only one degree of freedom,—that is to say, there is only one geometrical quantity or "generalized coordinate" by which the position of the system is determined,[11] which is to be found as a function of the time, —this single integral gives the complete solution of the problem, for everything then reduces to a differential equation of the first order with one independent and one dependent variable.

We have seen how a single differential equation of the second order between one independent variable (t) and one dependent variable (s) can be replaced by *two* simultaneous equations between one independent variable (t) and two dependent variables (s and v). Our system of $3n$ dynamical equations can be replaced by $6n$ equations of the first order connecting the independent variable with $6n$ dependent variables (the coordinates and the velocities).[12] This system has for its complete solution $6n$ integral equations connecting the variables and involving $6n$ constants of integration. In the case, for example, of the system of the dynamical equations, we can often so combine them that an integral results. Thus, in a more general case than that dealt with just above, from the system

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[10] Cf. *ibid.*, pp. 276-277, 343-344, 350.

[11] In the case of a rigid pendulum oscillating in a plane about one point, this quantity is obviously the angle a line fixed in the body and through that point makes with the vertical through that point.

[12] In the case of one or many particles, what can with advantage take the place of the X in our typical equation for *one* particle is a partial differential quotient of a certain function of the position of the system. This function (the "potential function") is a function of the coordinates of *all* the particles of the system, and is of very great importance in almost all parts of mathematical physics (cf. Mach, *Mechanics*, pp. 110, 398, 497-499).

$$m(d^2x/dt^2) = X, \quad m(d^2y/dt^2) = Y, \quad m(d^2z/dt^2) = Z,$$

we get, by multiplying the equations by dx, dy, and dz, respectively, and adding, the equation

$$m(dx/dt)d(dx/dt) + m(dy/dt)d(dy/dt) + m(dz/dt)d(dz/dt) = X.dx + Y.dy + Z.dz.$$

Now, if a force-function U exists which is a function of x, y, z, and not t, and such that the right-hand member is its exact differential, then both sides of the above equation are obviously exact differentials. Integrating, then, we get what older writers called the equation of *vis viva,* and nowadays we call the "equation of energy." Evidently this reasoning extends to cases of more than one particle.

We can also define a "first integral" of the system of $6n$ dynamical equations in an obviously equivalent way as follows: A first integral is an equation $\phi = \text{const.}$, where ϕ is a function of the $3n$ coordinates $(x, y, \ldots.)$, the $3n$ velocities $(v, w, \ldots.;$ where $v = dx/dt$, $w = dy/dt, \ldots.)$ and the time t, such that $d\phi/dt$ vanishes identically by means of the given system. Sometimes the function ϕ itself is called the integral. An integral is given by the principle of *vis viva*[13] just spoken of; and other integrals are given by the principles of the conservation of the center of gravity[14] and areas.[15] Let us consider more closely the conservation of the center of gravity.

We may consider the force X on the particle m in the direction of the x-axis to be composed of two parts: one (X′) due to the actions on m of the other particles of the system, and one (X″) due to the external forces on the system. We can thus put:

$$X = X' + X'',$$

and analogously for Y, Z, and so on. Now, by Newton's third law of motion, $\Sigma X'$, where the summation indicated by the Σ extends to all the particles of the system, is zero. If also there are no external forces, each X″ is zero. Thus, adding the equations of motion, we get:

$$\Sigma m(d^2x/dt^2) = 0,$$

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[13] Cf. Mach, Mechanics, pp. 343-350.

[14] *Ibid.,* pp. 287-293.

[15] *Ibid.,* pp. 293-305.

and two others for the y-coordinates and z-coordinates respectively. These equations can be at once integrated and give the "principle of the conservation of linear momentum" or of the conservation of the center of gravity. In fact, by one integration,

$$\Sigma m(dx/dt)=a,$$

where a is a constant. This is the theorem stated in the third Corollary to the Laws of Motion in the *Principia.* The fourth Corollary merely requires the knowledge that the coordinates of the center of gravity of the dynamical system are $\xi=\Sigma mx/\Sigma m$, and two other identities. Thus we see at once that $d\xi/dt$ is constant. The above equation expressing the conservation of momentum gives $3n$ first integrals, and $3n$ other integrals are obtained by renewed integrations. In the above typical instance, a second integration gives

$$\Sigma mx=at+b,$$

when b is a constant.

The case is not very dissimilar with the integrals expressing the conservation of "angular momentum" or of areas. In this principle we have $3n$ equations of the form

$$\Sigma m[x(d^2y/dt^2)-y(d^2x/dt^2)]=0,$$

which give, on integration, $3n$ first integrals of the form

$$\Sigma m[x(dy/dt)-y(dx/dt)]=c,$$

where c is a constant. But—and here the principle of areas differs from the principle of the conservation of linear momentum—no further integration in finite terms is possible.

Now d'Arcy's principle is the principle of areas, while the principle of least action is a condensed form of the equations of motion. The principle of least action is more general than that of areas, but the principle of least action does not give an integral. Consequently, we cannot allow that d'Arcy's claim[16] that his principle can replace what the principle of least action meant to state and that other integrals can be deduced from it, is valid. It might just as well be maintained that the knowledge of one root of, say, a cubic equation enables another to be calculated, in general.[17] In the case of another integral of the dynamical differential equations—the integral of *vis viva* just dealt with—the analogous problem has been, according to Boltzmann,[18] the vainly sought ideal of some of the supporters of the modern science of "energetics."

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[16] *Monist,* Vol. XXII, pp. 446, 453.

[17] If we are given a complex root, and we know that the coefficients of the equation are real, we know that another root must be conjugate; but this is not the general case.

II.

We shall here make two digressions. In the first place, even when the dynamical equations are under the most favorable conditions, so that all the above integrals subsist, we cannot, in general, solve the equations completely. Consider in the general case the motion of a system of particles which act on each other according to the law of gravitation. The acceleration of any particle P is compounded of accelerations to each of the others, and any one of the components thus arising is of the form $\gamma m/r^2$, where m is the mass of one of the particles and r is its distance from P. For such a system there would exist seven first integrals, and ten integrals in all, of the equations of motion. In fact the principle of the conservation of linear momentum would give six integrals in all, representing that the center of inertia of the system moves uniformly in a straight line; the principle of the conservation of moment of momentum would give three integrals, representing that the moment of momentum of the system about any one of the axes of reference is a constant quantity; the principle of the conservation of energy would also give an integral. The general problem of two bodies was solved by Newton in the *Principia*; but even in the case of three particles these ten integrals do not suffice for the complete description, in a finite form, of the motions of the system; and in 1887 Bruns[19] showed that all the other integrals of the problem of n bodies besides these algebraic functions of the coordinates and velocities are *transcendent*. The most interesting practical result of all this is that we cannot deduce from the law of gravitation an exact account of the motions of the bodies forming the solar system. But there are a number of circumstances which conduce to the possibility of deducing from this law such an approximate account of the motions in question as shall be sufficiently exact to agree with observation over a long period of time. Among these we may mention: (1) that the mass of the sun is great compared with that of the other bodies, even the mass of Jupiter being less than $\frac{1}{1000}$ part of that of the sun; (2) that all the orbits are nearly circular and lie nearly in one plane.

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[18] "Ein Wort der Mathematik an die Energetik," *Ann. der Phys. und Chem.*, Vol. LVII, 1896, pp. 39-71; cf. especially p. 40.

[19] "Ueber die Integrale des Vielkörper-Problems," *Leipziger Berichte*, Vol. XXXIX, 1887, pp. 1-39, 55-82. In this paper, Bruns seeks the algebraic integrals which do not contain t explicitly. Bruns also wrote another paper on the same subject in *Acta Mathematica*, Vol. XI, pp. 25 ff. Cf. A. R. Forsyth, *Theory of Differential Equations*, Vol. III, Cambridge, 1900, Chap. XVII; and E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, with an Introduction to the Problem of Three Bodies*, Cambridge, 1904, pp. 346-367.

In the second place, it is proper to refer here to the equations of dynamics as they appear when we use the vector methods of Hamilton, Grassmann, and others.[20] Instead of $3n$ equations, we have only n equations, and this gives us some idea of the shortness with which we can write out theorems. But the most important advantage of vector methods in mechanics and physics is that all the operations correspond exactly to the fundamental operations with motions of mechanics. Velocities and forces and axes of couples are various kinds of (unlocalized or localized) vectors; addition is like the composition of velocities; Grassmann's "inner" multiplication of two vectors is what, if one of the vectors is a force and the other the velocity of its point of application, we call "the rate of working of the force"; and so on. Then, as O'Brien[21] has remarked, the use of vector methods facilitates, in certain cases, the technical process of integration. Lastly, as I have pointed out in notes to my translation of Mach's *History and Root of the Principle of the Conservation of Energy*,[22] vector methods afford a powerful and effective means of discovering, and representing in intuitive form, the ultimate principles of physics.

III.

We will now return to the equations of mechanics. Since the system of the internal forces (such forces as we have denoted above by X') is equivalent to zero, it is convenient to be able to express the equations of motion in a form into which only the external forces (denoted by X'' above, and henceforward by X) enter. This was done by d'Alembert in 1743,[23] and a succinct expression in a variational form was given to d'Alembert's principle by Lagrange in 1788.[24] We will here merely state this formula, in which the whole of mechanics is condensed:

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[20] Mach, *Mechanics*, pp. 527-528, 577-579.

[21] *Phil. Trans.*, Vol. CXLII, 1851, pp. 161-206.

[22] Chicago, Open Court Publishing Co., 1911, pp. 106, 108.

[23] Mach, *Mechanics*, pp. 331-343.

[24] This is the date of the first edition of Lagrange's *Mécanique*; but the variational form of d'Alembert's principle appeared in papers of Lagrange's dating from 1764 and 1780.

$$\Sigma[m(d^2x/dt^2)\delta x + m(d^2y/dt^2)\delta y + m(d^2z/dt^2)\delta z] = \Sigma(X.\delta x + Y.\delta y + Z.\delta z)$$

and will refer to Mach's *Mechanics*[25] for the meaning of the δx's and so on used in it. Our purpose here is merely to indicate that the principle of least action is a transformation of this variational equation.[26]

We will first remark: (1) that this single equation contains, so to speak, the whole of mechanics; and (2) that it is far more general than the system of equations already dealt with, as it holds for a system subject to any conditions, even if they depend on the time, and does not presuppose the existence of a force-function.

When we differentiate by δ the equation of *vis viva,* substitute in the above general equation of mechanics, change the order of d and δ in such terms as $d\delta x$, integrate with respect to s, and determine that the variations of the coordinates vanish at the limits of integration, we obtain the result that

$$\delta\Sigma m \int u . ds = 0,$$

where u is the velocity of the particle m and ds is an element of the path described by that particle.

That the vanishing of the variation of a function implies that this function is a maximum or a minimum was never doubted by the older analysts, including Lagrange; but Sir William Rowan Hamilton replaced the name of "the principle of least action" by the less committal name of "the principle of stationary action."[27]

Conversely, by carrying out the variations in the equation last given and eliminating the variations δu by means of the equation of *vis viva* differentiated by δ, we arrive at the equation expressing that the integral with respect to t of the function which, when equated to zero, is the variational form of d'Alembert's principle, is zero. Now, this equation must subsist for all possible variations. and consequently the integrand must vanish at every moment.

Lagrange[28] remarked that the integral in the principle of least action transforms into

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[25] Mach, *Mechanics,* pp. 342-343, 466.

[26] Here we follow Lagrange (1788). A convenient annotated German translation of that part of Lagrange's *Mécanique* which refers to this point is given, with other matters, in No. 167 of *Ostwald's Klassiker.*

[27] The name of "principle of stationary action" is also often given to that other principle which is allied to the principle of least action, and is generally called "Hamilton's principle" (cf. for example, L. Boltzmann, *Vorlesungen über die Princípe der Mechanik,* Vol. II, Leipsic, 1904, p. 9).

[28] In the second edition of his *Mécanique* (Vol. I, 1811), not in the first (1788). Cf. *Œuvres de Lagrange,* Vol. XI, pp. 323-324; *Ostwald's Klassiker,* No. 167, pp. 11, 49.

$$\int 2T . dt,$$

and "therefore the principle in question reduces to this: the sum of the instantaneous *vires vivae* of all the bodies, from the moment when they start from given points to that when they arrive at other given points, is a maximum or a minimum. We might, then, call it, on better grounds, the *principle of the greatest* or *smallest vis viva,* and this way of regarding it would have the advantage of being general, both for motion and for equilibrium, since we have seen, in the third section of the first Part (art. 22), that the *vis viva* of a system is always a maximum or a minimum in a position of equilibrium."

In the second and subsequent editions of the *Mécanique,* articles 21-27[29] are under the heading: "Propriétés de l'équilibre, relatives aux maxima et minima," and correspond to articles 13-20 of the third section of the first part of the first edition.[30] In the 22d article of the second edition,[31] Lagrange proved, by using the equation of *vis viva,* Courtivron's principle that, of all the positions which the system successively takes, the one in which it has the greatest or the least *vis viva* is the same as that in which it must be placed in order that it may remain in equilibrium. Then Lagrange[32] proved that when the function considered is a minimum the equilibrium will be stable, and unstable if the function is a maximum. In the second edition Lagrange[33] gave a new proof of this theorem, and Dirichlet[34] simplified this proof and made it more rigorous.

Lagrange[35] mentioned that the principle that the function which he denoted by Π, and which is known as the "work-function" or the "force-function," is a maximum or a minimum in the case of equilibrium, is in what Maupertuis's "Loi de Repos" consists.

It must be remembered that as a matter of fact Maupertuis merely said that his function was a *minimum* for equilibrium, and he emphasized that the "Law of Rest" was only a special case of the principle of least action. It was essential for him to fix his attention solely on the minimum if he wished to prove that the Creator was economical, and not spendthrift. It was Euler who first drew the more[36] correct conclusion that the function is a maximum or a minimum and gave a correct sketch of the connection of the minimum with stability and the maximum with instability. These investigations have already been fully described,[37] and we need only indicate how Maupertuis's "Law of Rest" is related to the principle of least action.

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[29] *Œuvres,* Vol. XI, pp. 69-76.

[30] *Méchanique analitique,* Paris, 1788, pp. 36-44.

[31] *Œuvres,* Vol. XI, p. 70. This is exactly reproduced from the 15th article of the first edition (pp. 37-38), with the exception that, whereas the name of de Courtivron alone is cited in the first edition, in the second the Paris *Mémoires* of 1748 and 1749 alone are cited.

[32] *Méchanique,* pp. 38-44.

[33] *Œuvres,* Vol. XI, pp. 70-74.

[34] *Journ. für Math.,* Vol. XXXII, 1846, pp. 85-88; *Werke,* Vol. II, pp. 5-9; French translation in *Journ. de Math.,* Vol. XII, 1847; *Œuvres de Lagrange,* Vol. XI, pp. 457-459.

[35] *Méchanique,* p. 37 (cf. p. 11); *Œuvres,* Vol. XI, p. 69 (cf. p. 22).

The general variational principle (of Hölder) of which Hamilton's principle and the principle of least action are particular cases is[38]

$$\int\{2T.d\delta t+(\delta T+\delta U)dt\}=0$$

where the integral is to be taken between the limits t_0 and t_1, and the symbol "δU" does not imply that a force-function U exists.[39] In the statical case, $T=0$, so that we get

$$\int\delta U.dt=0.$$

As this holds for all virtual displacements (marked by "δ") we have simply

$$\delta U=0,$$

and this is the "Law of Rest."

IV.

Maupertuis's words:[40] "When some change happens in nature, the *quantity of action* [that is to say, 'the product of the mass of the bodies by their velocity and by the space which they describe,' a phrase which means Σmvs] necessary for this change is the smallest possible," has been interpreted to mean only changes of velocity. But this is not what Maupertuis plainly meant at first, in the case of the motion of light.[41] And if a body moves uniformly along the x-axis, there is change always going on; the velocity v or s/t is constant, and the total action in the space s described in time t is mvs or mv^2t. Again, if a mass m is under no forces, if we minimize the action $\int m.v.ds$, we find that $m.d^2x/dt^2=0$, and two similar equations in y and z. But, in his problems on impact, Maupertuis[42] changed his conception of "action" to one which seemed to suit his purpose better. This new conception was the quantity

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[36] Even in Euler's and Lagrange's works, the occurrence of neutral equilibrium is not considered. On the whole question, see Mach, *Mechanics,* pp. 68-73.

[37] *Monist,* Vol. XXII, pp. 421 (note 35) etc.

[38] *Ibid.,* p. 300.

[39] *Ibid.,* p. 290.

[40] *Monist,* Vol. XXII, p. 420.

[41] *Ibid.,* p. 418. Maupertuis's calculation of the path of light was, in spite of Mach's statement to the contrary, consistent with this (*ibid.,* pp. 285-286, 418-419, note 19).

$$A(a-x)^2+B(x-b)^2,$$

and so obviously the action of A, if its velocity were unaltered by impact, would be zero, whereas we have just seen that a body under inertial conditions has non-zero "action" according to Maupertuis's first definition.

Indeed it seems likely that d'Arcy[43] was right in conjecturing that Maupertuis merely chose his formula for the "action" so that, when minimized, it gave the results he wanted. In fact he did dishonestly, pretentiously, and unskilfully what Euler[44] did honestly, humbly, and skilfully.

Suppose, with Maupertuis,[45] that the inelastic A and B move with the respective velocities a and b in a straight line and in the same sense. Let A start behind B and catch it up, and then, after the impact, let both move with the common velocity, u, less than a and greater than b.

It must be remembered that Newton[46] had made a series of careful experiments on impact, the result of which may be expressed as follows: If u and u' are the velocities of the spherical masses m and m' in the line of their centers and in the same sense before impact, and v and v' their velocities in the same line and sense after impact, then

$$v-v'=-e(u-u'),$$

where e is a positive fraction which is independent of the masses and velocities of the bodies and is now called "the coefficient of restitution." The limiting cases, where e is zero or unity, were chiefly discussed in the eighteenth century: when $e=1$, the bodies are said to be "perfectly elastic"; when $e=0$, they are said to be "inelastic" or "hard." Very soft bodies, like putty or wool, are almost inelastic;

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[42] *Ibid.*, pp. 421, 446-447, 451, 452, 455. Cf. Mach's *Mechanics*, p. 368.

[43] *The Monist*, Vol. XXIII, p. 447. Bertrand's objections (*ibid.*, p. 456) are simply due to the fact that he misunderstood d'Arcy's (badly chosen) notation $P^2=\int \Phi.dx$, and so on.

[44] *Ibid.*, p. 437.

[45] *Ibid.*, p. 421.

[46] Scholium to the Laws of Motion in the *Principia*.

but so also obviously would a perfectly rigid body be. The epithet "hard" arose owing to the exclusive consideration of rigid bodies as inelastic. Problems about the impact of elastic and inelastic bodies were then in the focus of interest. This was partly due to the emphasis laid on the principle of *vis viva* by Huygens and Leibniz.[47] In the case of impact, this principle only holds when $e=1$.

Returning to Maupertuis's problem, at a certain instant A and B both start from given points. Both A and B move towards the origin. Let us suppose that after impact the motion continues with the common velocity u, constant or not. We have to find u. In Maupertuis's principle we consider the whole class of motions obtained by giving u different values, so that the mass consisting of A and B together reach the origin at different times in the members of the class of motion considered. Of this class of motions we have to seek out that motion which the "action" is the least. Keeping to Maupertuis's first definition of "action," we can put the problem into the variational form

$$\delta\Sigma m\int v.ds=0,$$

or

$$\Sigma m\int(\delta v.dx+v.d\delta x)=0$$

where v is dx/dt. As there are no external forces, $\delta U=0$, and so the terms $m.\delta v.dx$ are zero. Integrating by parts, we get

$$\Sigma m\int dv.\delta x=0,$$

and, since δx is arbitrary,

$$\Sigma m.dv=0,$$

or Σmv is constant. Thus, for inelastic masses

$$u(A+B)=Aa+Bb$$

while, for elastic masses, α and β are given by

$$A\alpha+B\beta=Aa+Bb,\ \beta-\alpha=a-b.$$[48]

With regard to Maupertuis's treatment of the lever,[49] it is to be noticed that the forces on A and B were not explicitly considered by Maupertuis. This is the principal defect[50] in his formulation. In the cases of impact considered there were no external forces; but, if there are external forces, we must assume that they are expressible in terms of the masses and velocities by the principle of *vis viva* or its generalization. What we need, as Lagrange showed, is the equation $\delta T = \delta U$; and a sufficient condition for this is the equation $T = U + h$. The equation of *vis viva* is not a necessary condition for the subsistence of the equation $\delta T = \delta U$, as I have often remarked above.[51] But without the condition $\delta T = \delta U$ the principle of least action is meaningless.

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[47] *Monist,* Vol. XXII, pp. 424-425; Mach, *Mechanics,* pp. 317, 326, 274-276.

[48] *Monist,* Vol. XXII, pp. 422, 447 (note 152).

[49] *Monist,* Vol. XXII, pp. 422, 448, 452-453, 455.

[50] Mach, *Mechanics,* p. 366.

It seems to me very possible that, in this consideration about the lever, Maupertuis was wishing to assert Courtivron's theorem[52]—a deduction from Maupertuis's "Law of Rest"—that the *vis viva* was a maximum when the system passed through a position of stable equilibrium. At any rate, Maupertuis later[53] emphasized that the principle of least action reduced, in the statical case, to the "Law of Rest." Euler mentioned[54] the problem of the lever with approval, and he could hardly have gone so far as this if it did not seem capable of a valid interpretation.

It is not easy to see how Maupertuis could have reasonably maintained this without granting that the principle of *vis viva* is to hold. And yet this would unfit the principle of least action for dealing with the impact of inelastic bodies. This difficulty must have presented itself to Euler, but he did not mention it. It was Lagrange[55] who solved it, and showed how $\delta\Sigma\int m.v.ds = 0$ had a mechanical meaning: v was a function of s, even though we cannot say that $T = U + h$. But Lagrange seems to have forgotten his advance, and it only showed itself again in Hölder's work.

Maupertuis is, then, right; but he gave no valid reason in support of his thesis. He was unpardonably obscure, and not always, perhaps, unwilling to pretend that he saw truths where he did not but would have liked to. D'Arcy criticized Maupertuis rather destructively, and d'Arcy was often right. Where he was wrong Maupertuis did not see, except in one thing: the definition of "action."[56]

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[51] *Monist*, Vol. XXII, pp. 457, 289-290.

[52] See Section III above.

[53] *Monist*, Vol. XXII, pp. 420-421, note 35.

[54] *Ibid.*, p. 436.

[55] *Ibid.*, pp. 289-290.

[56] *Ibid.*, pp. 445, 450, 454.